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# Some estimates for the curvatures of complete spacelike hypersurfaces in generalized Robertson–Walker spacetimes

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**Abstract**

In this paper we obtain some estimates for the higher order mean curvatures, the scalar curvature and the Ricci curvature of a complete spacelike hypersurface in a generalized Robertson–Walker spacetime, under certain assumptions on the warped function of the ambient space. Our results will be an application of a generalized maximum principle due to Omori.


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*Keywords:* Generalized Robertson–Walker spacetimes; Spacelike hypersurface; *j*th mean curvatures; Scalar curvature; Ricci curvature; Generalized maximum principle

## 1. Introduction

In this paper we study spacelike hypersurfaces in the family of cosmological models known as generalized Robertson–Walker (GRW) spacetimes. GRW spacetimes are warped products of a (negative definite) universal time as a base and a Riemannian manifold as a fiber (see Section 2), and they extend classical Robertson–Walker spacetimes to include the cases in which the fiber does not have constant sectional curvature (we refer the reader to [9, Chapter 7] to get an introduction to warped products). GRW spacetimes include, for

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instance, the de Sitter spacetime, the Friedmann cosmological models and the static Einstein spacetime.

GRW spacetimes are suitable spacetimes to model universes with inhomogeneous spacelike geometry [10]. In fact, it is well-known that conformal changes of the metric of a GRW spacetime with a conformal factor which only depends on $t$, produce new GRW spacetimes. Even more, small deformations of the metric on the fiber of Robertson–Walker spacetimes also fit into the class of GRW spacetimes. Thus, a GRW spacetime is not necessarily spatially homogeneous, as in the classical cosmological models. Recall that spatial homogeneity seems appropriate just as a rough approach to consider the universe in the large, but not to consider it in a more accurate scale, because this assumption could not be realistic.

In this paper we establish some a priori estimates for the curvatures of complete spacelike hypersurfaces in a GRW spacetime $\bar{M}$ which are contained in certain unbounded regions of the ambient space determined by suitable assumptions on the warped function of $\bar{M}$. Our study is motivated by the paper [1], where the first author jointly with Alías established several estimates for the curvatures of a complete hypersurface in the de Sitter space (see also [5,7]).

By *curvatures* here we mean the higher order mean curvatures of the hypersurface, as well as its scalar and Ricci curvatures. Let us recall that the $j$th mean curvatures $H_j$, for $j = 1, \ldots, n$, are the natural generalization of the mean and scalar curvatures of the hypersurface, and they are defined, up to a constant, by the elementary symmetric functions of the principal curvatures. It follows from the Gauss equation of the hypersurface that $H_j$ is extrinsic when $j$ is odd and its sign depends on the chosen orientation, while $H_j$ is intrinsic when $j$ is even.

Our results will be an application of the following generalized maximum principle for Riemannian manifolds given by Omori [8] (see also Yau's paper [12]).

*A generalized maximum principle*: Let $M$ be a complete Riemannian manifold whose sectional curvatures are bounded away from $-\infty$ and let $u \in C^2(M)$ be a function bounded from above. Then, for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M$ such that

(i) $|\nabla u(p_\varepsilon)| < \varepsilon$,
(ii) $(\nabla^2 u)_{p_\varepsilon}(v, v) < \varepsilon$, for all tangent vector $v \in T_pM$, $|v| = 1$,
(iii) $\sup u - \varepsilon < u(p_\varepsilon) \leq \sup u$,

where $\nabla u$ and $\nabla^2 u$ denote, respectively, the gradient and the Hessian of $u$.

## 2. Preliminaries

Let $(F, g)$ be an $n$-dimensional ($n \geq 2$) Riemannian manifold and let $I \subset \mathbf{R}$ be an open interval in $\mathbf{R}$ endowed with the metric $-\mathrm{d}t^2$. The warped product $\bar{M} = I \times_f F$ endowed with the Lorentzian metric

$$\langle , \rangle = \pi_I^*(-\mathrm{d}t^2) + f^2(\pi_I)\pi_F^*(g),$$

where $f > 0$ is a smooth function on $I$, and $\pi_I$ and $\pi_F$ denote the projections onto $I$ and $F$ respectively, is said to be a *generalized Robertson–Walker* (GRW) spacetime with *base* $(I, -\mathrm{d}t^2)$, *fiber* $(F, g)$ and *warping function* $f$ (see [2]).

A smooth immersion $\psi : M \to \bar{M}$ of an $n$-dimensional connected manifold $M$ is said to be a *spacelike hypersurface* if the induced metric via $\psi$ is a Riemannian metric on $M$, which, as usual, is also denoted by $\langle , \rangle$.

Note that the timelike vector field $\partial_t = \partial/\partial t \in X(\bar{M})$ determines a time-orientation on $\bar{M}$. Thus, if $\psi : M \to \bar{M}$ is a spacelike hypersurface, we can put in each point $p \in M$

$$\partial_t(p) = \partial_t^\top(p) + \partial_t^\perp(p),$$

where $\partial_t^\top \in X(M)$ and $\partial_t^\perp$ is a non-vanishing vector field normal to $M$. Then we will take $N$ as the vector field which results by normalizing $\partial_t^\perp$, which is a timelike unit normal vector field on $M$ in the same time-orientation that $\partial_t$, that is, verifying that $\langle \partial_t, N\rangle \le -1$. We will refer to $N$ as the *Gauss map* of $M$.

In order to set up the notation to be used later, we will denote by $\bar{\nabla}$ and $\nabla$ the Levi-Civita connections of $\bar{M}$ and $M$, respectively. Then the Gauss and Weingarten formulas for $M$ in $\bar{M}$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y - \langle A(X), Y\rangle N \tag{1}$$

and

$$\bar{\nabla}_X N = -A(X) \tag{2}$$

for all tangent vector fields $X, Y \in X(M)$, where $A$ stands for the shape operator of $M$ in $\bar{M}$ with respect to the Gauss map $N$ (see [9, Chapter 4]).

Associated to the shape operator of $M$ there are $n$ algebraic invariants, which are the elementary symmetric functions $\sigma_r$ of its principal curvatures $k_1, \ldots, k_n$, given by

$$\sigma_j(k_1, \ldots, k_n) = \sum_{i_1<\cdots<i_j} k_{i_1}, \ldots, k_{i_j}, \quad 1 \le j \le n.$$

The $j$th mean curvature $H_j$ of the spacelike hypersurface is then defined by

$$\binom{n}{j} H_j = (-1)^j \sigma_j(k_1, \ldots, k_n) = \sigma_j(-k_1, \ldots, -k_n).$$

When $j = 1$, $H_1 = -(1/n)\,\mathrm{tr}(A) = H$ is the mean curvature of $M$. The choice of the sign $(-1)^j$ in our definition of $H_j$ is motivated by the fact that in that case the mean curvature vector is given by $\vec{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in M$ if and only if $\vec{H}(p)$ is in the time-orientation determined by $N(p)$. On the other hand, when $j = n$, $H_n = (-1)^n \det(A)$ defines the Gauss–Kronecker curvature of the spacelike hypersurface.

The spacelike slices $F(t_o) = \{t_o\} \times F$, $t_o \in I$, will play an important role in our work. As it can be easily seen, $F(t_o)$ has shape operator $A = (-f'(t_o)/f(t_o))I_n$, so that $F(t_o)$ is a totally umbilical hypersurface of $\bar{M}$ with constant $j$th mean curvature

$$h_j = \left(\frac{f'(t_o)}{f(t_o)}\right)^j.$$

## 3. Main results

The classical maximum principle allows us to obtain the following estimates for the higher order mean curvatures of a spacelike hypersurface $\psi : M \to \bar{M}$ in a GRW spacetime when the function $u = \pi_I \circ \psi$ is bounded on $M$ and $f'$ has a suitable sign on the hypersurface. In what follows, we write $f'(u) \gg 0$ (resp. $f'(u) \ll 0$) to express that there exists a positive (resp. negative) constant $k$ such that $f'(u) \geq k > 0$ (resp. $f'(u) \leq k < 0$).

**Theorem 1.** *Let $\bar{M} = I \times_f F$ be a GRW spacetime and $\psi : M \to \bar{M}$ a complete spacelike hypersurface whose sectional curvatures are bounded away from $-\infty$.*

(a) *If $f'(u) \gg 0$ on $M$ and there exists $\alpha = \inf(u)$, then*

$$\sup H_j \geq \left(\frac{f'(\alpha)}{f(\alpha)}\right)^j.$$

(b) *If $f'(u) \ll 0$ on $M$ and there exists $\beta = \sup(u)$, then*

$$\sup H_j \geq \left(\frac{f'(\beta)}{f(\beta)}\right)^j$$

*whenever $j$ is even, and*

$$\inf H_j \leq \left(\frac{f'(\beta)}{f(\beta)}\right)^j$$

*whenever $j$ is odd.*

**Proof.** From the Gauss formula (1) it is not difficult to see that the gradient of $u$ is $-\partial_t^\top$, where

$$\partial_t^\top = \partial_t + \langle \partial_t, N\rangle N \in X(M) \tag{3}$$

denotes the tangential component of $\partial_t$. On the other hand, using (1) and (2), we also obtain that the Hessian of $u$ is given by

$$\nabla^2 u(X, Y) = -\frac{f'(u)}{f(u)}(\langle X, Y\rangle + \langle X, \partial_t^\top\rangle\langle Y, \partial_t^\top\rangle) + \langle N, \partial_t\rangle\langle AX, Y\rangle \tag{4}$$

for all tangent vector field $X, Y \in X(M)$.

(a) Since $u$ is a smooth function on $M$ bounded from below by $\alpha = \inf(u)$, we know from the generalized maximum principle that for each $\varepsilon_m = 1/m$ there exists a point $p_m \in M$ such that

$$|\nabla u(p_m)| < \varepsilon_m, \qquad (\nabla^2 u)_{p_m}(v, v) > -\varepsilon_m \tag{5}$$

for all tangent vector $v \in T_{p_m}M$, $|v| = 1$, and

$$\alpha + \varepsilon_m > u(p_m) \geq \alpha.$$

Let $\{e_i^m\}_{i=1,\dots,n}$ be an orthonormal basis of principal directions at the point $p_m$ satisfying $A_{p_m}(e_i^m) = k_i(p_m)e_i^m$. We obtain from (5) and (4) that

$$(\nabla^2 u)_{p_m}(e_i^m, e_i^m) = -\frac{f'(u(p_m))}{f(u(p_m))}(1 + \langle e_i^m, \partial_t^\top(p_m)\rangle^2) + \langle N(p_m), \partial_t(p_m)\rangle k_i(p_m) > -\frac{1}{m}$$

and, since $\langle N(p_m), \partial_t(p_m)\rangle < 0$, it follows that

$$k_i(p_m) < \frac{1}{\langle N(p_m), \partial_t(p_m)\rangle}\left(\frac{-1}{m} + \frac{f'(u(p_m))}{f(u(p_m))}(1 + \langle e_i^m, \partial_t^\top(p_m)\rangle^2)\right). \quad (6)$$

Since $f'(u) \gg 0$ on $M$, $k_i(p_m)$ must be negative for $m$ sufficiently large. Let us assume from now on that $m$ is large enough so that $k_i(p_m) < 0$. Note that, from (3), we get

$$-1 = \langle \partial_t^\top, \partial_t^\top\rangle = |\nabla u|^2 - \langle \partial_t, N\rangle^2$$

so that

$$-\langle \partial_t(p_m), N(p_m)\rangle = \sqrt{1 + \nabla u(p_m)^2} < \sqrt{1 + \varepsilon_m^2}.$$

Then we have

$$\binom{n}{j} H_j(p_m) > \binom{n}{j}\left(\frac{-1}{\sqrt{1+\varepsilon_m^2}}\left(\frac{-1}{m} + \frac{f'(u(p_m))}{f(u(p_m))}(1 + \langle e_i^m, \partial_t^\top(p_m)\rangle^2)\right)\right)^j$$

and letting $m \to \infty$ it follows that

$$\sup H_j \geq \left(\frac{f'(\alpha)}{f(\alpha)}\right)^j.$$

(b) From a similar argument, for each $\varepsilon_m = 1/m$ there exists a point $q_m \in M$ such that

$$|\nabla u(q_m)| < \varepsilon_m, \qquad (\nabla^2 u)_{q_m}(v, v) < \varepsilon_m$$

for all tangent vector $v \in T_{q_m}M$, $|v| = 1$, and

$$\beta - \varepsilon_m < u(q_m) \leq \beta.$$

An analogous reasoning allows us to obtain

$$k_i(q_m) > \frac{1}{\langle N(q_m), \partial_t(q_m)\rangle}\left(\frac{1}{m} + \frac{f'(u(q_m))}{f(u(q_m))}(1 + \langle e_i^m, \partial_t^\top(q_m)\rangle^2)\right),$$

being $k_i(q_m)$ positive for $m$ sufficiently large. Hence we have:

- If $j$ is even, letting $m \to \infty$ it follows that

$$\sup H_j \geq \left(\frac{f'(\beta)}{f(\beta)}\right)^j.$$

- If $j$ is odd, letting $m \to \infty$ it results

$$\inf H_j \leq \left(\frac{f'(\beta)}{f(\beta)}\right)^j. \qquad \square$$

**Remark 2.** Note that, in each case, the bound for the curvature $H_j$, $j = 1, 2, \ldots, n$, is the curvature $h_j$ of the slice which determines the region where the hypersurface is contained.

We can obtain similar estimates for the curvatures of a spacelike hypersurface such that the function $u$ acquires a local maximum or minimum on $M$ and $f'$ has a suitable sign at such points:

**Theorem 3.** *Let $\psi : M \to \bar{M}$ be a spacelike hypersurface in a GRW spacetime $\bar{M} = I \times_f F$.*

(a) *If $u$ attains a local minimum at a point $p_1 \in M$ and $f'(u(p_1)) \geq 0$, then*

$$\sup H_j \geq \left(\frac{f'(u(p_1))}{f(u(p_1))}\right)^j.$$

(b) *If $u$ attains a local maximum at a point $p_2 \in M$ and $f'(u(p_2)) \leq 0$, then*

$$\sup H_j \geq \left(\frac{f'(u(p_2))}{f(u(p_2))}\right)^j,$$

*whenever $j$ is even, and*

$$\inf H_j \leq \left(\frac{f'(u(p_2))}{f(u(p_2))}\right)^j,$$

*whenever $j$ is odd.*

**Proof.** Since $u$ attains a local minimum at $p_1$ we have that $\nabla u(p_1) = -\partial_t^\top = 0$ and $(\nabla^2 u)_{p_1}$ is positive semidefinite. A similar argument as in Theorem 1 allows us to obtain the result.

The proof of (b) is analogous. $\square$

From now on, we will study the case of a spacelike hypersurface $M$ in a GRW spacetime $\bar{M}$ with constant sectional curvature $\bar{c}$. As is well known (see [3]), under this assumption the fiber $F$ has constant sectional curvature, that is, $\bar{M}$ is a classical Robertson–Walker spacetime. Under this additional hypothesis, the Ricci curvature of $M$ is given by

$$\mathrm{Ric}(X, Y) = \bar{c}(n-1)\langle X, Y\rangle - \mathrm{tr}(A)\langle A(X), Y\rangle + \langle A(X), A(Y)\rangle \qquad (7)$$

for $X, Y \in X(M)$, and the second higher order mean curvature $H_2$ is, up to a constant, the scalar curvature $S$ of M; indeed

$$S = n(n-1)(\bar{c} - H_2).$$

The spacelike slices $F(t_o)$ of such spacetimes have constant Ricci curvature

$$\mathrm{ric} = (n-1)\left(\bar{c} - \left(\frac{f'(t_o)}{f(t_o)}\right)^2\right)\langle , \rangle$$

and constant scalar curvature

$$s = n(n-1)\left(\bar{c} - \left(\frac{f'(t_o)}{f(t_o)}\right)^2\right).$$

It is possible to give the following estimates for the Ricci and scalar curvatures of complete spacelike hypersurfaces in GRW spacetimes of constant sectional curvature. In what follows, we will denote by $UM = \{(p, v) : p \in M, v \in T_pM, |v| = 1\}$ the unit sphere bundle of $M$.

**Theorem 4.** *Let $\bar{M} = I \times_f F$ be a GRW spacetime of constant sectional curvature $\bar{c}$ and $\psi : M \to \bar{M}$ a complete spacelike hypersurface whose sectional curvatures are bounded away from $-\infty$.*

(a) *If $f'(u) \gg 0$ on $M$ and there exists $\alpha = \inf(u)$, then*

$$\inf_{(p,v)\in UM} \mathrm{Ric}_p(v, v) \le (n-1)\left(\bar{c} - \left(\frac{f'(\alpha)}{f(\alpha)}\right)^2\right)$$

*and*

$$\inf S \le n(n-1)\left(\bar{c} - \left(\frac{f'(\alpha)}{f(\alpha)}\right)^2\right).$$

(b) *If $f'(u) \ll 0$ on $M$ and there exists $\beta = \sup(u)$, then*

$$\inf_{(p,v)\in UM} \mathrm{Ric}_p(v, v) \le (n-1)\left(\bar{c} - \left(\frac{f'(\beta)}{f(\beta)}\right)^2\right)$$

*and*

$$\inf S \le n(n-1)\left(\bar{c} - \left(\frac{f'(\beta)}{f(\beta)}\right)^2\right).$$

**Proof.** Under the assumptions of (a), we obtain from (7) and (6) that

$$\begin{aligned}\mathrm{Ric}(e_j^m, e_j^m) &= \bar{c}(n-1) - \sum_{i=1}^{n} k_i(p_m)k_j(p_m) + k_j^2(p_m)\\ &= \bar{c}(n-1) - \sum_{i\neq j} k_i(p_m)k_j(p_m) \le \bar{c}(n-1) - (n-1)\\ &\quad\cdot\left(\frac{1}{\langle N(p_m), \partial_t(p_m)\rangle}\left(\frac{-1}{m} + \frac{f'(u(p_m))}{f(u(p_m))}(1 + \langle e_i^m, \partial_t^\top(p_m)\rangle^2)\right)\right)^2\end{aligned}$$

and letting $m \to \infty$ we get that

$$\inf_{(p,v)\in UM} \mathrm{Ric}_p(v,v) \le (n-1)\left(\bar{c} - \left(\frac{f'(\alpha)}{f(\alpha)}\right)^2\right).$$

Bearing in mind that $S = \mathrm{tr}(\mathrm{Ric})$, it follows the result on the scalar curvature.

The case (b) is analogous. □

**Remark 5.** Again, the bounds for the Ricci and scalar curvatures of the hypersurface $M$ are, respectively, the curvatures ric and $s$ of the slice which determines the region where $M$ is contained.

We also have the following theorem.

**Theorem 6.** *Let $\psi : M \to \bar{M}$ be a spacelike hypersurface in a GRW spacetime $\bar{M} = I \times_f F$ of constant sectional curvature $\bar{c}$.*

(a) *If $u$ attains a local minimum at a point $p_1 \in M$ and $f'(u(p_1)) \ge 0$, then*

$$\inf_{(p,v)\in UM} \mathrm{Ric}_p(v,v) \le (n-1)\left(\bar{c} - \left(\frac{f'(u(p_1))}{f(u(p_1))}\right)^2\right)$$

*and*

$$\inf S \le n(n-1)\left(\bar{c} - \left(\frac{f'(u(p_1))}{f(u(p_1))}\right)^2\right).$$

(b) *If $u$ attains a local maximum at a point $p_2 \in M$ and $f'(u(p_2)) \le 0$, then*

$$\inf_{(p,v)\in UM} \mathrm{Ric}_p(v,v) \le (n-1)\left(\bar{c} - \left(\frac{f'(u(p_2))}{f(u(p_2))}\right)^2\right)$$

*and*

$$\inf S \le n(n-1)\left(\bar{c} - \left(\frac{f'(u(p_2))}{f(u(p_2))}\right)^2\right).$$

## 4. Conclusions and final comments

Spacelike hypersurfaces in a spacetime play an important role in General Relativity. As is known, they can be used as initial surfaces to solve the Cauchy problem in spacetimes. Specifically, on those hypersurfaces which have constant mean curvature, the constraint equations split into a linear system and a non-linear elliptic equation (see [4], for instance). On the other hand, three-dimensional relative spatial universes of certain observers in spacetimes may be modeled from spacelike hypersurfaces. Recall also that several Singularity

Theorems assumed the existence of a spacelike hypersurface, with some physically reasonable assumptions on its second fundamental form [9, Chapter 14]. Indeed, the second fundamental form contains all the information to predict the time evolution in future and in past of spatial universes.

In a GRW spacetime $\bar{M}$, the integral curves of $\partial_t$ are called *comoving observers*. This family of natural observers is clearly distinguished among other ones. The timelike vector field $f\partial_t$ represents an a priori symmetry of the GRW spacetime $\bar{M}$ and is conformal. This gives $\mathrm{div}(\partial_t) = (n-1)f'/f$, where div is the divergence with respect to the Lorentz metric of $\bar{M}$. Thus, if $f' > 0$ holds, then $\mathrm{div}(\partial_t) > 0$, which indicates that the comoving observers are, on average, spreading apart. Similarly, $f' < 0$ says that the comoving observers come together [11, p. 121]. Relative spatial universes of comoving observers are the spacelike slices $t =$ constant. On the other hand, the unit normal vector field $N$ on a spacelike hypersurface $M$ extends locally to a reference frame $Z$ (its integral curve through $p \in M$ is the geodesic $s \mapsto \exp_{\psi(p)}(sN_p)$ of $\bar{M}$). We have proposed in this paper to relate observable quantities on $M$ for the observers in $Z$ and the corresponding ones on certain spacelike slices for the observers in $\partial_t$.

Essentially, we have used two natural assumptions: the first one about the warping function, restricted on the spacelike hypersurface $M$, which may be interpreted as an expansion/contraction of the spacetime for comoving observers. The second assumption is a regularity behavior about the curvature of $M$, which is automatically satisfied if $M$ is compact, and hence complete, in a (necessarily, see [2]) spatially closed spacetime $\bar{M}$. We have obtained several inequalities relating the $j$th mean curvature (resp. Ricci curvature) of $M$ and the $j$th mean curvature (resp. Ricci curvature) of suitable spacelike slices of the GRW spacetime $\bar{M}$.

In order to emphasize the physical application of our results, note that, under our assumptions, Theorem 1 reads

(a) If comoving observers of $\bar{M}$ are, on average, spreading apart on $M$, then every $j$th mean curvature $H_j$ of $M$ satisfies

$$\sup H_j \geq h_j(\alpha),$$

where $\alpha$ is the infimun value of the restriction to $M$ of the universal time $t$, and $h_j$ is the $j$th mean curvature of the spacelike slice $t = \alpha$.

(b) If comoving observers of $\bar{M}$ are, on average, coming together on $M$, then

$$\sup H_j \geq h_j(\beta) \quad (\text{resp.} \inf H_j \leq h_j(\beta)),$$

where $\beta$ is the supremun value of the restriction to $M$ of the universal time $t$, and $h_j$ is the $j$th mean curvature of the spacelike slice $t = \beta$ and $j$ is even (resp. $j$ is odd).

On the other hand, if $\bar{M}$ is indeed assumed a classical Robertson–Walker spacetime, Theorem 4 says

If comoving observers of $\bar{M}$ are, on average, spreading apart (resp. coming together) on $M$, then the Ricci curvature Ric of $M$ satisfies

$$\inf \mathrm{Ric} \leq \mathrm{ric}_\alpha \quad (\text{resp.} \inf \mathrm{Ric} \leq \mathrm{ric}_\beta),$$

where $\alpha$ and $\beta$ are as previously, Ric is considered as a function on the unitary tangent bundle of $M$ and $\mathrm{ric}_\alpha$ (resp. $\mathrm{ric}_\beta$) is the (constant) Ricci curvature of the spacelike slice $t = \alpha$ (resp. $t = \beta$).

It is normally argued that the curvature of a realistic spacetime must satisfy a suitable *energy condition* which expresses the presence of matter and/or electromagnetic radiation in spacetime. Note that the GRW spacetimes in Theorems 4 and 6 obey the *null energy condition* (which only says presence of radiation) because both cases assume constant sectional curvature. The most used energy condition is the so called *timelike convergence condition* (TCC) which says $\overline{\mathrm{Ric}}(v, v) \geq 0$, for all timelike tangent vector $v$ (sometimes, it is said that TCC is the mathematical translation that gravity, on average, attracts). For instance, in [2,6], spacelike hypersurfaces of constant mean curvature in GRW spacetimes obeying TCC (and the stronger energy condition $\overline{\mathrm{Ric}}(v, v) > 0$, for all timelike tangent vector $v$) have been studied, from different approaches. So, it naturally appears to carry out our study on $j$th mean curvatures to spacelike hypersurfaces in GRW spacetimes satisfying TCC. On the other hand, it would be of interest to consider analogous problems on conformally stationary (CS) spacetimes (i.e. spacetimes which admits a timelike conformal vector field $\xi$). This family contains properly the one of GRW spacetimes. Specially those CS spacetimes such that $\xi$ is not irrotational deserves to be contemplated to that end.

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